



Dynamically equilibrium shapes and directions of motion of ocean current rings[☆]

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ABSTRACT

The dynamically equilibrium shapes of a uniform-density rotating mass of liquid (a ring) in the surface layer of a quiescent stratified ocean are determined. The examination is carried out in a plane tangential to the Earth, taking into account the vertical and horizontal projections of the angular velocity of its rotation. Exact solutions of the equations of motion of an ideal incompressible fluid are obtained, making it possible, for a linearly stratified ocean, to determine the dynamic all equilibrium shape of the interfaces of water masses and the free boundaries of cyclonic and anticyclonic rings. These shapes comprise second-order surfaces inclined to the water level in the meridian plane, the type of surfaces depending on the governing parameters of the problem. Expressions are obtained for the angles of inclination of the principal axes. For small deviations from equilibrium, due to a difference in the gravitational forces and Archimedes forces, motion of the ring occurs, governed by the inclination of the principal axes and the nature of change (increase or reduction) in the average density of the ring, determined by the ratio of the rates of diffusion of heat and salt. The displacement along the parallel comprises geostrophic motion, for the velocity of which an analytical expression is obtained. The displacement along the meridian comprises motion over an inclined plane. An analytical expression is given that relates the change in the depth of the centre of mass of the ring to the velocity of motion along the meridian through the angle of inclination of the principal axes of the ring. This explains the motion of both types of Gulf Stream ring to the south-west and of the Oyasio ring to the north-east.

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Rings are near-surface mesoscale (horizontal dimensions $\sim 10^2$ km) eddies formed as a result of the closure of loops of oceanic flows.^{1–3} The most well-known rings are the Gulf Stream and Kuroshio. These currents pass along the interfaces of water masses that differ significantly in temperature, salinity and density.

The density distributions of water over its depth on both sides of the Gulf Stream, plotted from available data,⁴ indicate that the water density to the north-west of the Gulf Stream (cold water) and the water density to the south-east of the current (warm water) have initial sections with a near-linear dependence on depth. Here, at identical depths, the density of the cold water is appreciably greater than the density of the warm water. Thus, the density of warm water at a depth of about 600 m corresponds to the density of cold water at a depth of 200 m. From depths of ~ 1500 m, the densities of cold and warm water are similar and hardly change with depth.

The generation of rings by the looping of currents is the principal mechanism of transfer of water masses across frontal zones. Full-scale observations indicate that, for the Gulf Stream and Kuroshio, rings formed to the north of the currents are anticyclonic (rotating clockwise), while rings to the south of the currents are cyclonic. The anticyclonic rings of the Gulf Stream have an isothermal heat core of significant size that contains water of the Sargasso Sea, while cyclonic rings have a cold core of captured water to the north-west of the current.^{2,3} The water density in the region of the Gulf Stream depends chiefly on the temperature and depends very little on the salinity. Essential for a long lifetime and for the direction of motion of the ring as a whole, in our opinion, is the difference in its average density from the density of the surrounding liquid. This is determined by the core of lower density for the anticyclonic ring and of greater density for the cyclonic ring. For simplicity, we will henceforth assume that the rings comprise rotating formations of uniform density.

Like submerged mesoscale intrusion eddies ('lenses'),^{5,6} surface eddies (rings) can be regarded as natural fluid gyroscopes. A water mass rotating relative to the Earth has an intrinsic angular momentum, the vector of which rotates together with the Earth (precesses), and thus changes its orientation in absolute space. This precession is possible only under the action of a moment of external forces. In an ideal model (without dissipation), the moment of forces ensuring this change in orientation of the angular momentum comprises the moment of hydrostatic forces in the stratified ocean, acting on the dynamically equilibrium shape of the rotating liquid mass that is formed.

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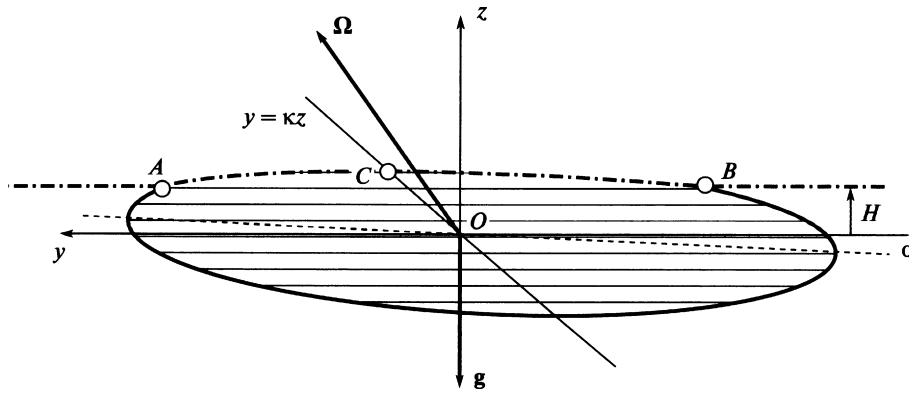


Fig. 1.

Rings can exist for a fairly long time. The anticyclonic rings of the Gulf Stream live for several months, and the cyclonic rings for two or more years. Gulf Stream rings of both types, moving slowly in the ocean, preferentially to the south-west, travel hundreds of kilometres.^{1,2,7} Their long life seems to be due, in particular, to the existence under actual conditions of a shape similar to the dynamically equilibrium shape of the ideal model.⁸ Eddy formations not having a dynamically equilibrium shape, for example cyclonically rotating lenses of Mediterranean water in the Atlantic, do not last long. Therefore, only rings having dynamically equilibrium shapes are considered below.

1. The hydrodynamic problem

A uniform-density rotating mass of an ideal incompressible fluid (a ring) is submerged in a stratified ocean that is quiescent relative to the Earth rotating at an angular velocity Ω . In view of the smallness of the horizontal dimensions of the rings compared with the radius of the Earth, our examination will be carried out in a plane layer, tangential to the Earth, of stratified ideal fluid in a plane-parallel gravitational field. The dynamically equilibrium shape of the interface of water masses is constructed from the condition for the pressures at the interface to be equal for the submerged part and for the pressure at the boundary of the ring with the atmosphere to be constant. The latter condition and also the examination of cyclonic rings distinguish this paper from the investigation conducted earlier.⁶

We will introduce a Cartesian coordinate system. The z axis is directed upwards along a local vertical, the plane $z=0$ lies at depth where the density of the ocean is equal to the density of the ring. The x axis is directed to the east, and the y axis to the north.

To describe the steady motion, we will use the equations of hydrodynamics in the Gromeka–Lamb form in a coordinate system connected to the rotating Earth

$$\text{div } \mathbf{V} = 0$$

$$(2\Omega + \text{rot}\mathbf{V}) \times \mathbf{V} = -\frac{1}{\rho} \text{grad}\Phi, \quad \Phi = p + \frac{1}{2}\rho V^2 + \rho g z \tag{1.1}$$

where \mathbf{V} is the relative velocity of the liquid, ρ is the density, p is the pressure, Ω is the angular velocity of rotation of the Earth and g is the acceleration due to gravity.

We will specify projections of the relative velocity of the liquid of the ring in the form

$$u = -\omega(y - \kappa z), \quad v = \omega x, \quad w = 0, \quad \omega = \text{const} \tag{1.2}$$

Such a velocity field corresponds to planar motions relative to the Earth in horizontal circles, the centres of which are positioned on the line $y = \kappa z$ in the meridional plane with an angular velocity ω identical for all horizontal sections. The point O of intersection of this line with the plane $z=0$ will be taken as the origin of coordinates. The meridional sections of anticyclonic and cyclonic rings, passing through point O , are shown schematically in Figs. 1 and 2. Here, the dash-dot line shows the ocean surface, ABC shows the elevation (lowering) of the ocean surface, C shows the top (bottom) point, $z=H$ is the plane of the undisturbed ocean surface, the continuous horizontal lines show the lines of intersection of the planes of rotation of the liquid with the meridional plane and the dashed line shows one of the principal axes of the ellipsoid. The vectors of the angular velocity of rotation ω are applied at the centres of the circular cross-sections, i.e., at points of the line $y = \kappa z$. They are directed vertically downwards in Fig. 1 and vertically upwards in Fig. 2.

By a direct check we can ascertain whether the equation of continuity is satisfied.

The equations of motion (1.1) take the form (ρ_r is the ring density)

$$\begin{aligned} \omega x(2\Omega_z + 2\omega) &= \frac{1}{\rho_r} \frac{\partial \Phi}{\partial x} \\ \omega(y - \kappa z)(2\Omega_z + 2\omega) &= \frac{1}{\rho_r} \frac{\partial \Phi}{\partial y} \\ -\omega(y - \kappa z)(2\Omega_y + \kappa\omega) &= \frac{1}{\rho_r} \frac{\partial \Phi}{\partial z} \end{aligned} \tag{1.3}$$

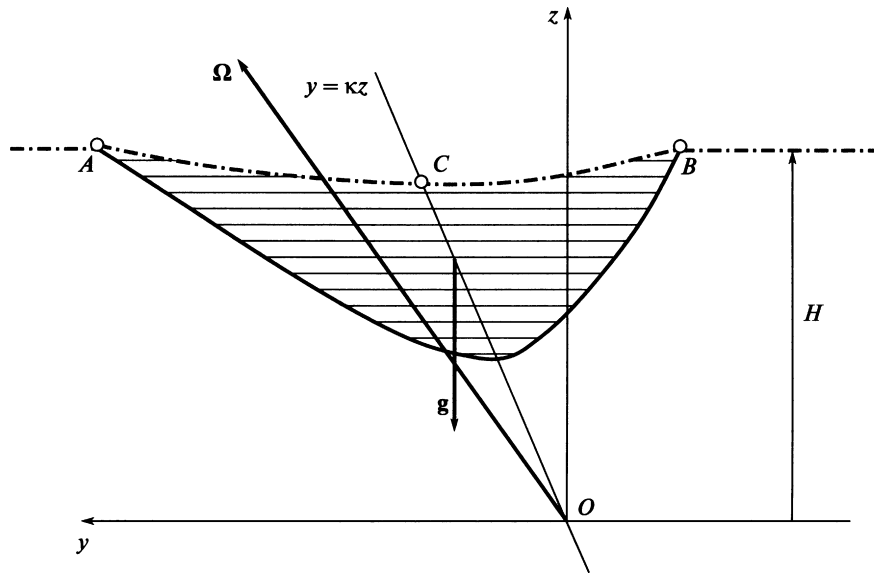


Fig. 2.

The consistency condition for the first two equations of system (1.3) is satisfied automatically. From the requirements of consistency of the first and third equations, and also the second and third equations, the following condition must be satisfied by the tangent of the angle of inclination of the line of the centres of rotation to the vertical κ

$$\kappa = 2\Omega_y / (\omega + 2\Omega_z) \tag{1.4}$$

The projections of a relative eddy are $(0, \omega\kappa, 2\omega)$, the projections of a planetary eddy are $(0, 2\Omega_y = 2\Omega \cos \varphi, 2\Omega_z = 2\Omega \sin \varphi)$; φ is latitude), and the projections of an absolute eddy are $(0, 2\Omega_y + \kappa\omega, 2\Omega_z + 2\omega)$.

It turned out that the parameter κ , introduced as a characteristic of the geometrical location of the centres of the circular sections, has a profound physical significance, namely, the slope of the vector of an absolute eddy to the local vertical is equal to κ .

System (1.3) is used to determine the pressure in the ring. The pressure in the outer region is found from the equations of hydrostatics for a stratified fluid. We will determine the shape of the interface of water masses from the condition for the pressures on it to be equal.

Integration of system of equations (1.3) gives the pressure distribution in the ring:

$$p_r = \rho_r F(x^2 + y^2, y, z) - \rho_r g z + P$$

$$F = \omega \frac{\omega + 2\Omega_z}{2} (x^2 + y^2) + \omega \frac{2\Omega_y^2}{\omega + 2\Omega_z} z^2 - 2\Omega_y \omega y z \tag{1.5}$$

We determine the integration constant P from the condition of zero pressures of the atmosphere and ring on the circumference of intersection of the ring and the plane surface $z=H$ of the quiescent ocean. The coordinates of point A (Figs. 2 and 3) are $x=0, y=\kappa H+R, z=H$, where R is the radius of the indicated circumference. From Eq. (1.5) we have

$$P = \rho_r g (H - h_u), \quad h_u = \frac{\omega}{2g} (\omega + 2\Omega_z) R^2 \tag{1.6}$$

and the expression for the pressure in the ring takes the form

$$p_r = \rho_r F(x^2 + y^2, y, z) + \rho_r g (H - z - h_u) \tag{1.7}$$

On the free ring boundary, the pressure is zero, i.e., the upper boundary is the zero isobar, the coordinates of which are x_u, y_u, z_u . From relation (1.7) it follows that

$$F(x_u^2 + y_u^2 - R^2, y_u, z_u) - g(z_u - H) = 0 \tag{1.8}$$

For $\omega \neq 0$, the surface (1.8) is an elliptical paraboloid. Three cases of positioning of the paraboloid are possible:

- Case 1 (Case 2): when $\omega < 0$ and $\omega + 2\Omega_z > 0$ – a weak anticyclone, i.e. $|\omega| < 2\Omega_z$ (when $\omega < 0$ and $\omega + 2\Omega_z < 0$, i.e. $|\omega| > 2\Omega_z$ – a strong anticyclone); in this case, $z_u > H$ ($z_u < H$), i.e. the points of the free boundary are positioned higher (lower) than the undisturbed surface of the ocean – an elevation (depression);
- Case 3: when $\omega > 0$, a cyclone; in this case $z_u < H$, i.e. the points of the free boundary are positioned below the undisturbed surface of the ocean – a depression.

For known values of ω and R , the quantity h_u (1.6) defines the depth of the depression or the magnitude of the elevation. For rings, the characteristic values $R \sim 50\text{--}100$ km and the peripheral velocities $v_r \sim 1$ m/s, so that $\omega \sim 2 \times 10^{-5} \text{ s}^{-1}$. At latitudes of $30\text{--}35^\circ$,

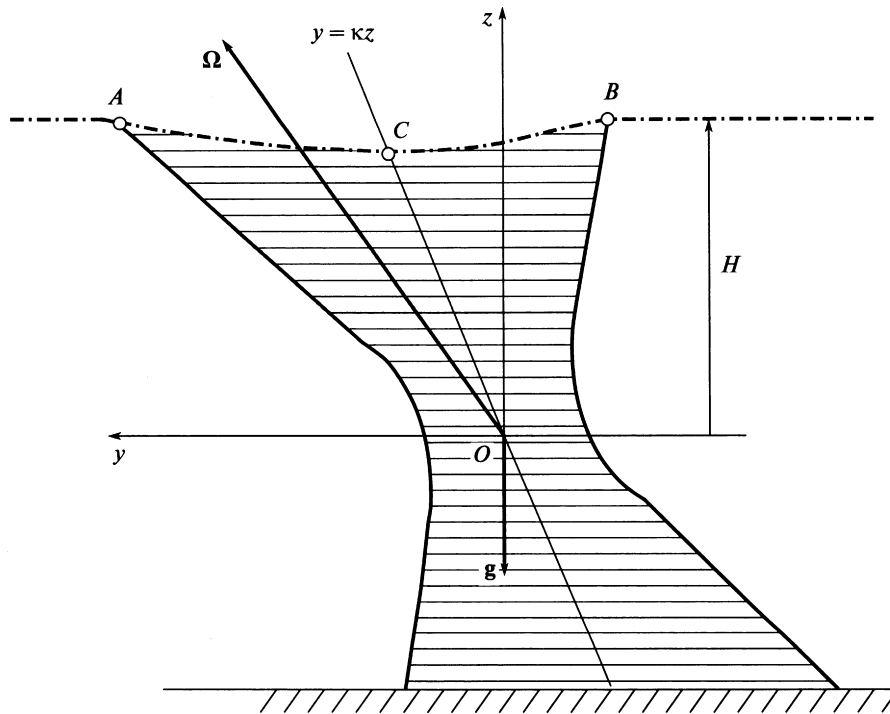


Fig. 3.

where the Gulf Stream rings exist, $\Omega_z \sim 4 \times 10^{-5} \text{ s}^{-1}$, and therefore the depth of the depression is $\sim 25\text{--}45 \text{ cm}$ for cyclonic rings and the elevation is $\sim 15\text{--}35 \text{ cm}$ for anticyclonic rings. The boundary case separating strong anticyclones from weak ones corresponds to the condition $|\omega| = 2\Omega_z$. Here, the peripheral velocity for the characteristic dimensions and latitudes of location of rings would be $v_r \approx 2 \text{ m/s}$. It seems that strong anticyclonic rings in the ocean are not observed because the velocities in rings are lower than the value indicated.

Observations from satellites indicate that anticyclonic rings have elevations with a height of the order of 10 cm or more,⁹ and cyclonic rings have depressions with a depth of the same order. Thus, the model of a ring reflects and explains the results of these observations.

We will write the equation of an elliptical paraboloid (1.8) in canonical form in the variables x_u, y_u, z_u :

$$\frac{x_u^2}{a^2} + \frac{(y_u'')^2}{b^2} = \pm z_u'', \quad a^2 = 4(1 + \kappa^2)b^2 = \frac{2g\kappa}{|\omega|\Omega_y(1 + \kappa^2)^{1/2}}$$

A plus sign is taken for a cyclonic ring, and a minus sign for an anticyclonic ring. Quantitative estimates with the parameter values indicated above give, for a cyclonic (anticyclonic) ring, $a \approx 110 \text{ (120) km}$, $b \approx 36 \text{ (27) km}$, $\text{arctg } \kappa \approx 50^\circ \text{ (63}^\circ\text{)}$.

One of the principal axes of the paraboloid is parallel to the line $y = \kappa z$, while the variables y_u'' and z_u'' are related to the variables y_u and z_u by explicit relations.

The free surface in the form of an elliptical paraboloid is formed by circular streamlines for an anticyclonic ring or by circular annular streamlines for a cyclonic ring in each horizontal section. Here, in an anticyclonic ring the elevation is displaced to the north of the centre of mass of the ring, and in a cyclonic ring the depression is displaced, and the point of application of Archimedes forces turns out to be different from the centre of mass of the ring. This creates around the centre of mass of the ring a moment of the pair of gravitational and Archimedes forces, directed to the west for an anticyclonic ring and to the east for a cyclonic ring. This moment ensures precession of the ring together with the rotating Earth.

When determining the interface of a ring and the water surrounding it, we will assume that the stratification of the density ρ_f of the surrounding water is linear in z in the region of occurrence of the ring, right up to the free surface:

$$\rho_f = \rho_r + \left. \frac{d\rho_f}{dz} \right|_{z=0} z$$

The hydrostatic pressure p_f in the water surrounding the ring is determined from the equations of hydrostatics. At the interface of the water masses ($x = x_b, y = y_b, z = z_b$) we have

$$p_f(z_b) = \rho_r g(H - z_b) + \frac{d\rho_f}{dz} g \frac{H^2 - z_b^2}{2} \tag{1.9}$$

Equating the right-hand sides of expressions (1.7) and (1.9), we find the equation of the interface of the water masses

$$F(x_b^2 + y_b^2, y_b, z_b) - \frac{1}{2} N^2 z_b^2 = -\frac{\Delta}{\rho_r} \tag{1.10}$$

where

$$\Delta = \rho_r \left[\frac{1}{2} N^2 H^2 - g h_u \right], \quad N^2 = -\frac{d\rho_f g}{dz \rho_r}$$

N^2 is the square of the Väisälä–Brunt frequency for a stably stratified fluid $d\rho_f/dz < 0$.

An analysis of the invariants of quadratic form on the left-hand side of Eq. (1.10) indicates that, in weak anticyclonic rings ($\omega < 0$, $|\omega| < 2\Omega_z$), the interface of the water masses is an ellipsoid. In cyclonic rings ($\omega > 0$) under the conditions

$$\Delta > 0 \tag{1.11}$$

$$\left(\sigma + \frac{\zeta}{2} - N^2 \right) > 0 \quad \text{или} \quad \left(\sigma + \zeta - N^2 \right) < 0; \quad \sigma = \frac{2\omega\Omega_y^2}{\omega + 2\Omega_z}, \quad \zeta = \omega(\omega + 2\Omega_z) \tag{1.12}$$

the interface is a parted hyperboloid. At the latitudes of formation of cyclonic rings of the Gulf Stream ($\approx 30^\circ$), with actual values of the parameters (at average depths of $\sim 5 \times 10^2$ m) of the arrangement of the rings of the Gulf Stream, the density of the surrounding water $\rho_f \sim 27$ nominal units (in units of nominal density $\sigma_t = [\rho_t(T, s, p_a)]/\rho_t(4, 0, p_a) - 1] \times 10^3$, $\rho_t(4, 0, p_a) = 1 \text{ g/cm}^3$ is the density of water at 4°C and normal pressure p_a , and s is the salinity), $d\rho_f/dz \sim 1.3 \times 10^{-3}$ nominal units/m, $N^2 \sim 1.3 \times 10^{-5} \text{ s}^{-2}$, $\omega \sim 2 \times 10^{-5} \text{ s}^{-1}$, $\Omega_z \approx 3.6 \times 10^{-5} \text{ s}^{-1}$, $R/H \sim 10^2$), the condition (1.11) is satisfied. The second condition of system (1.12) is satisfied with a large margin: the final term in the brackets is three orders of magnitude greater than the previous terms.

The interface at $\Delta < 0$ is an unparted hyperboloid (this can occur at high angular velocities of rotation, e.g. in a typhoon (see below in this section)); when $\Delta = 0$ it is a cone.

Bringing the quadratic form on the left-hand side of equation (1.10) to a canonical form in the variables y'_b and z'_b by rotating the axes by an angle α about the tangent to the parallel (the x axis), we find

$$\text{tg}2\alpha = \frac{2\kappa}{\kappa^2 - 1 - N^2\kappa(2\Omega_y\omega)^{-1}} \tag{1.13}$$

For actual parameters of rings, the final term in the denominator is five orders of magnitude greater than the first two. Therefore, $\text{tg}2\alpha \approx -4\Omega_y\omega/N^2$. Since $4\Omega_y|\omega| \ll N^2$, the angle α is small and

$$\alpha \approx -2\Omega_y\omega/N^2 \tag{1.14}$$

The exact expressions for the coefficients at x_b^2, y_b^2, z_b^2 in canonical form in terms of the governing parameters are unwieldy, and we will therefore use the fact that the angle α is small. Omitting the calculations, we will give the equation of the interface in the principal axes with approximate values of the coefficients under the condition that $R^2 - N^2H^2\kappa(2\Omega_y\omega)^{-1} = \chi \neq 0$:

$$\frac{x_b^2 + y_b'^2}{a^2} + \frac{z_b'^2}{c^2} = 1; \quad a^2 = \chi, \quad c^2 = H^2 - R^2 2\Omega_y\omega (N^2\kappa)^{-1} \tag{1.15}$$

For an anticyclonic ring ($\omega < 0$), the exact equation of the interface is the equation of a triaxial ellipsoid, and the approximate interface (1.15) is an ellipsoid of revolution, the principal axes of which are inclined in the meridian plane such that the southern edge of the ellipsoid is submerged, declining by a small angle α from the interface. For actual parameters of the ring, $\alpha \approx 40$ seconds of arc.

For a cyclonic ring ($\omega > 0$), the exact equation of the interface is the equation of a triaxial hyperboloid, and the approximate surface (1.15) is a hyperboloid of revolution, the principal axes of which are inclined to the meridian plane by a small angle α to the north. When $\chi < 0$, the hyperboloid is a parted hyperboloid, and with the opposite inequality it is an unparted hyperboloid. For actual parameters of cyclonic rings, the first inequality is satisfied, and the interface of the ring waters and the surrounding waters comprises the upper part of the parted hyperboloid (Fig. 2).

When $\chi = 0$, the interface is a cone with its apex at the origin of coordinates.

Estimates of the actual combination of parameters in Eq. (1.15) indicate that rings in the form of unparted hyperboloids do not exist in the ocean.

Unlike the ocean, in the atmosphere, for high angular velocities of the air ($\omega > 2\Omega_z$), for example in a typhoon, the dynamically equilibrium shape may be an unparted hyperboloid. We will make a rough estimate of the parameters of a typhoon. The height of the troposphere, where a typhoon propagates, is ~ 10 km, the air density distribution can be assumed to be roughly linear with height, and $N \sim 10^{-2} \text{ s}^{-1}$.^{2,10} We will assume the homogeneous core of the typhoon to be equal to the average density of the troposphere, which corresponds to a height $H \sim 5$ km. On the upper boundary of the troposphere, we will assume a pressure of zero. The angular velocity vectors of horizontal sections are directed vertically upwards. Assuming that the peripheral velocities in a typhoon are ~ 50 m/s, $R \sim 100$ km, we obtain $\omega \sim 5 \times 10^{-4} \text{ s}^{-1}$, $\omega \approx 7\Omega_z$, and inequality (1.11) changes sign. The second inequality of system (1.12) is satisfied with a large margin. Analysis indicates that the boundary of the homogeneous core of the typhoon is an unparted hyperboloid, which corresponds to the schematic model of a typhoon.^{11,12} However, this hyperboloid is non-axisymmetrical about the vertical. For the latitude $\varphi \sim 25^\circ$ with the above parameters of the typhoon, the centre line of the circular sections is deflected from the vertical in the meridional plane by about 13° (Fig. 3). With these simplifications and the indicated parameters of the typhoon, from the second equation of system (1.6), taking into account the fact that, for the characteristic latitudes ($\sim 25^\circ$) at which hurricanes exist in the Atlantic, $\omega \gg 2\Omega_z$, we obtain a depth of the isobar depression on

the upper boundary of the typhoon $h_u \sim \omega^2 R^2 / (2g) \sim 130$ m. For hurricane Katrina, in which the velocity reached 315 km/h, the quantity h_u amounts to 390 m.

This solution is a three-dimensional non-axisymmetric solution. In each horizontal section, the velocities depend linearly on the radius, i.e., the motion is that of a rigid body. However, on the whole this motion is not rigid-body motion. A rotating rigid body immersed in an ocean on the rotating Earth, to ensure precession of its angular momentum in absolute space, would be inclined towards the local water level. Here, the necessary moment of hydrostatic forces is created in the stratified medium, and the plane of rotation becomes non-horizontal.⁶ To create the same moment of forces, the rotating fluid mass adapts by changing its shape, leaving the plane of rotation horizontal and the angular momentum vertical. This is the essential difference in the mechanisms of adaptation to dynamic equilibrium of a rotating rigid body and a rotating fluid mass.

Note that *a posteriori* estimation of the viscous effects indicate that the viscosity forces in the case considered are equal to zero due to the linear dependence of the velocities on the coordinates, and the dissipative function has a single non-zero term $2\mu\omega^2\kappa^2$ (μ is the coefficient of dynamic viscosity). It is many orders of magnitude less than the magnitude of heat flow by heat conduction that occurs when there is a temperature difference between the core of the ring and the surrounding fluid. This indicates the negligibly small influence of viscosity in the region of the homogeneous core of the ring, which promotes its long life.

The solution of the problem of the equilibrium shape of a homogeneous rotating mass of liquid in a stratified rotating ocean with a free boundary obtained above is exact within the framework of the formulation used, because, at the interface, through the pressure continuity condition, two exact solutions of the equations of an ideal fluid are matched. This solution is discontinuous, as, at the interface of the fluids, the density and the velocity tangential to the surface have a discontinuity. The question of the structure of the interface requires a separate consideration.

Unlike to long-lived submerged mesoscale oceanic eddies ('lenses'), having a closed dynamically equilibrium shape only in the case of anticyclone rotation,^{5,6} near-surface mesoscale eddies of the same nature (rings) have dynamically equilibrium shapes also with cyclonic rotation, by deformation of the free surface in the form of a depression, during which the forces governed by the pressure gradient are directed inwards across the radius and balance the centrifugal forces of inertia and the Coriolis forces directed outwards.

The solutions obtained enable us to find expressions for the hydrostatic moments acting on rings and that arise on account of the non-coincidence of the centres of application of the gravitational forces and Archimedes forces created by the stratified fluid forced out by the ring. Unlike submerged lenses, which do not have a free surface, the asymmetrical deformation of the free surface that occurs in the case of near-surface eddies, depending on the direction of rotation, intensifies the hydrostatic moments and may even become decisive.

2. Determination of the directions of motion of rings

It is well known that both anticyclonic and cyclonic rings of the Gulf Stream are displaced slowly to the south-west. Normally, the western component of the velocity vector of translational motion is due to the influence of the so-called β -effect (a linear approximation of the dependence of the vertical component of the velocity of rotation of the Earth on the latitude).¹³ However, this kinematic effect cannot be the cause of the velocity of the ring as a whole along the parallel and does not explain the eastern velocity component of the motion of Oyasio anticyclonic eddies. The meridional velocity component of the motion of rings of the Gulf Stream has not been explained.¹³ Below, a dynamic mechanism is proposed that explains the motion of Gulf Stream rings towards the south-west and the motion of Oyasio anticyclonic eddies towards the north-east.⁹

Thermochaline processes (heat and salt diffusion from the ring into the surrounding water, and in the opposite direction) lead to a change in the average density of the ring, and, depending on the predominance of a particular type of diffusion, cause the submersion or surfacing of the ring, and, more accurately, a change in depth of its centre of mass.

Accelerations of the centres of mass of the rings with respect to the Earth are very small ($\sim 10^{-7}$ g). Therefore, when analysing the motion of the centre of masses, the motion of the closed liquid formations is replaced by the motion of a solid inclined circular disc. This enables us to use the well-developed hydrodynamic theory of added masses, taking into account the factors specific to geophysical hydrodynamics – stratification of the medium and the rotation of the Earth. The problem reduces to investigating a system of ordinary differential equations, the coefficients of which take into account these specific features of the problem. Here, from the hydrodynamic problem examined above, use is made of the dimensions (the radius of the disc is much greater than its thickness) and slope α of the principal axes of inertia of such a disc in the meridian plane. The motion of a solid in an ideal stratified liquid on the rotating Earth leads to trapping perturbations of the liquid in a finite region limited by a radius of the order of the Rossby deformation radius (the Rossby radius) $R_* = \sqrt{gH}/f_0$, where H_r is the characteristic vertical dimension (in the present case, the height of the ring) and $f_0 = 2\Omega \sin \varphi$ is the Coriolis parameter,^{10,14} and by a height of the order of the Rossby height $h_* = f_0 L / N$, where L is the characteristic horizontal dimension.¹⁴

Estimates of the added masses during motion of a body in a fluid depend on the flow scheme adopted. To estimate the added mass during motion in a direction perpendicular to the plane of a circular disc of radius R_d , we will examine its motion along a vertical, to be specific, downwards. The adopted flow scheme is depicted in Fig. 4. The disc is horizontal, because, to estimate the added mass, with the motion examined, the small inclination of the disc to the water level is negligible. The velocity of motion of the disc along the vertical, on account of the large added mass, because of the smallness of the motive force, equal to the difference in gravitational force and the Archimedes forces (the densities of the ring and the surrounding water are similar), and because of the large force of resistance to vertical motion, is very small (according to estimates, $\sim 10^{-6}$ m/s). Therefore, the surface of the ocean can be assumed to be undisturbed and can be replaced by a solid wall. Water flowing out from beneath the disc travels to the sides of the disc. Under the action of Coriolis forces, the velocity of particles of liquid flowing out from beneath the disc deviates from the radial direction, so that the radial velocities are transformed into peripheral velocities at a finite radius of the order of the Rossby radius R_* . The depth of penetration of the disturbances is limited by a height of the order of the Rossby height h_* .¹⁴ For the characteristic parameters of the rings of ocean currents, $H_r \sim 500$ m, $2R = L \sim 100$ km, $N \sim 0.4 \times 10^{-2} \text{ s}^{-1}$ and $\varphi \approx 30^\circ$, we have $R_* \approx 1000$ km and $h_* \approx 2000$ m. In the scheme, we show the velocity w of motion of the disc downwards along the vertical and the boundaries of the flow zone: the surface of the ocean (the dash-dot line), the depth to the upper boundary of the disc h_w , the side wall of the cylinder, which has a radius equal to the Rossby radius R_* , and the lower boundary, a distance of the Rossby height h_* from the disc. The shaped arrows show the directions of the cross-section-average velocities of motion

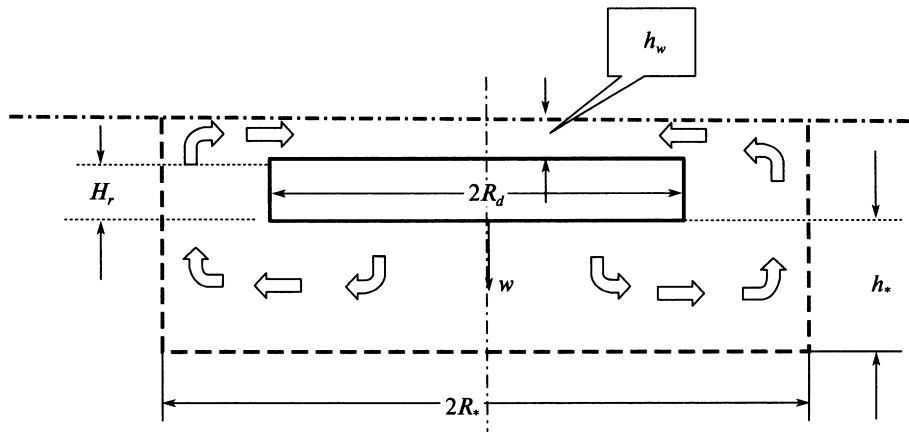


Fig. 4.

of the water. The added masses are estimated on the basis of an approximate calculation of the kinetic energy of masses of liquid with the selected flow scheme. Here, since the Coriolis forces of inertia, which convert the radial motion of the liquid into rotating motion, do not perform mechanical work, the kinetic energy of rotational motion is not taken into account. The subsequent discussion is based on relations of equal flow rates. Stratification is not taken into account, as the difference in density from a constant is very small and has practically no effect on estimates of the added masses.

For downward motion of the disc along the vertical at velocity w , the average velocity V_1 of discharge from beneath the disc over the Rossby height is determined from the condition of equality of the flow rates $\pi R_d^2 w dt = 2\pi R_d h_* V_1 dt$:

$$V_1 = \frac{R_d}{2h_*} w \tag{2.1}$$

Assuming the distribution of the horizontal component of the velocity of particles beneath the ring during such a discharge to be linear over the radius, we find the kinetic energy T_1 of particles of the liquid that flow out from beneath the ring (here and below, we assume that the density $\rho = 1$):

$$T_1 = \frac{1}{2} \int_0^{2\pi} \int_0^{R_d} r d\phi h_* dr \left(\frac{rw}{2h_*} \right)^2 = \frac{\pi w^2 R_d^4}{16h_*}$$

Particles flowing out from beneath the disc “push apart” particles lying outside the zone beneath the ring at the same depth. We will assume that the distribution of the radial components of the velocity of particles lying outside the zone beneath the ring decreases linearly along the radius, such that these components vanish over a finite radius – the Rossby radius R_* :

$$V_2(r) = \frac{V_1(R_d)}{R_* - R_d} (R_* - r)$$

The corresponding part of the kinetic energy T_2 , using Eq. (2.1), is defined by the expression

$$T_2 = \frac{\pi}{4h_*} \left(\frac{R_d w}{R_* - R_d} \right)^2 \left[\frac{R_*^4}{12} - R_d^2 \left(\frac{R_*^2}{2} - \frac{2}{3} R_* R_d + \frac{R_d^2}{4} \right) \right]$$

From the condition of mass balance of the fluid flowing out from beneath the ring and flowing into the cylindrical region between the side surfaces of the ring and cylinder with the Rossby radius ($\pi R_d^2 w = \pi (R_*^2 - R_d^2) v_v$), we find the average vertical component of the velocity of particles of the fluid

$$v_v = \frac{R_d^2 w}{R_*^2 - R_d^2}$$

The same distribution of vertical velocities will exist on the upper boundary of this cylindrical region. The kinetic energy of the liquid in the region indicated is

$$T_3 = \frac{\pi}{2} \frac{R_d^4 w^2}{R_*^2 - R_d^2} H_r$$

The flow rate of fluid flowing into the zone above the disc will be the same as that of liquid flowing out from the zone beneath it: $Q = \pi R_d^2 w$, but the thickness h_w of the layer into which this fluid is flowing is much less than the thickness h_* of the layer from which the fluid flows out from beneath the disc and much less than the disc radius. We determine the velocity of flow v_{in} into the upper zone from the equality of flow rates $2\pi R_d h_w v_{in} = \pi R_d^2 w$. It is equal to $v_{in} = R_d w / (2h_w)$. Since $R_d \gg h_w$, then $v_{in} \gg w$.

In a ring, $0 < h < h_w$, $R_d < r < R_*$, the fluid moves inwards along the radius. We will assume here that the distribution of the radial velocities is linear along the radius with the conditions $u(R_*) = 0$ and $v(R_d) = -R_d w / (2h_w)$. We obtain

$$v(r) = \frac{R_d w}{2h_w(R_* - R_d)}(r - R_*)$$

The kinetic energy of the fluid in this region is as follows:

$$T_4 = \frac{\pi R_d^2 w^2}{4h_w(R_* - R_d)^2} \left(\frac{1}{12} R_*^4 + \frac{2}{3} R_* R_d^3 - \frac{1}{2} R_*^2 R_d^2 - \frac{1}{4} R_d^4 \right)$$

We will assume the distribution of the horizontal velocity components of fluid particles over the disc to be linear: $v_u = -rw / (2h_w)$. The kinetic energy corresponds to this motion:

$$T_5 = \pi h_u \int_0^{R_d} \left(\frac{rw}{2h_w} \right)^2 r dr = \frac{\pi w^2}{16h_w} R_d^4$$

We will find the ratios \hat{T}_i of the components T_i ($i = 1, 2, \dots, 5$) to the kinetic energy of the disc:

$$T_d = \frac{\pi}{2} R_d^2 H_r w^2$$

We obtain

$$\hat{T}_1 = \frac{R_d^2}{8h_* H_r}, \quad \hat{T}_3 = \frac{R_d^2}{R_*^2 - R_d^2} \approx \frac{R_d^2}{R_*^2}, \quad \hat{T}_5 = \frac{R_d^2}{8h_w H_r}$$

$$h_* \hat{T}_2 = h_w \hat{T}_4 = \frac{1}{2H_r} \left(\frac{1}{R_* - R_d} \right)^2 \left[\frac{R_*^4}{12} - R_d^2 \left(\frac{R_*^2}{2} - \frac{2}{3} R_* R_d + \frac{R_d^2}{4} \right) \right] \approx \frac{R_*^4}{24H_r}$$

The approximate equalities are written taking into account the condition $R_* \gg R_d$.

It is obvious that the result depends strongly on the selected parameter h_w . We will assume that it is equal to 30 m – this is the depth of the upper layer of a real ocean where the density is almost constant. We obtain

$$\hat{T}_1 \approx 3.1 \cdot 10^2, \quad \hat{T}_2 \approx 4.5 \cdot 10^4, \quad \hat{T}_3 \approx 2.5 \cdot 10^{-3}, \quad \hat{T}_4 \approx 3 \cdot 10^6, \quad \hat{T}_5 \approx 2 \cdot 10^4 \quad (2.2)$$

i.e., the ratio of the overall kinetic energy of the liquid to the kinetic energy of the disc is $\sim 3 \times 10^6$. From this it follows that the added mass of the liquid λ_{33} during motion of the disc perpendicular to its plane is $\sim 3 \times 10^6$ times greater than its mass.

It is obvious that the parameters \hat{T}_4 and \hat{T}_5 increase without limit as $h_w \rightarrow 0$ and become decisive when estimating of the added mass λ_{33} .

To assess the added mass of the ring during its motion in the horizontal plane at a velocity W_h , we will model the motion of the ring as motion in the midplane of a circular disc of radius R_d . We will assume that the motion of the liquid occurs only in the horizontal planes (its stratification promotes this). On a circumference of Rossby radius R_* , we assume that the radial velocities are zero. The projection of the velocity onto the normal to the side surface of the disc is equal to $W_h \cos \theta$, where θ is the angle between the direction of velocity W_h and the normal. We will assume the distribution of radial velocities of the liquid to be linear, such that on the Rossby radius this component of the velocity is equal to zero:

$$W_r(r, \theta) = W_h \frac{R_* - r}{R_* - R_d} \cos \theta \quad (2.3)$$

The square of the average radial velocity over the length of section $R_* - R_d$ is equal to

$$\langle W_r(\theta) \rangle^2 = W_h^2 \frac{R_d^2}{(R_* - R_d)^2} \cos^2 \theta \quad (2.4)$$

Consider the sector of the disc with semi-aperture angle θ . The entire volume of the water displaced by this sector in unit time (H_r is the height of the disc)

$$V_s(\theta) = 2 \int_0^\theta W_h \cos \theta R_d H_r d\theta = 2 W_h R_d H_r \sin \theta$$

flows out through segments of the side walls of the sector of length $R_* - R_d$. This flow rate is equal to $2 \langle W_o \rangle H_r (R_* - R_d)$, where $\langle W_o \rangle$ is the average velocity of discharge through the side surfaces. From the equality of the flow rates we find

$$\langle W_o(\theta) \rangle = W_h \sin \theta \frac{R_d}{R_* - R_d} \quad (2.5)$$

Using expression (2.4) and the square of expression (2.5), for the total kinetic energy of the liquid enclosed between the disc and a cylinder of Rossby radius we obtain

$$T_h = \frac{\pi}{2} W_h^2 H_r R_d^2 \frac{R_* + R_d}{R_* - R_d} \approx \frac{\pi}{2} W_h^2 H_r R_d^2 \tag{2.6}$$

(we used the relation $R_* \gg R_d$), i.e., the kinetic energy of the liquid is roughly equal to the kinetic energy of the disc. From this it follows that the associated mass of liquid during motion of the disc in its plane is roughly equal to the mass of the disc itself.

Estimates of the added masses show that $\lambda_{33} \sim 3 \times 10^6 M$, and the added masses in other directions are roughly equal to the mass of the body M :

$$\lambda_{11} \approx M, \quad \lambda_{22} \approx M \tag{2.7}$$

Relations (2.7) refine the estimates of λ_{11} and λ_{22} given earlier.⁵

The approximate equations of motion of the centre of masses of such a disc,⁵ taking into account the refined estimates (2.7), have the form

$$\begin{aligned} (M + \lambda_{11}) R_e \cos \varphi d^2 \lambda / dt^2 - [(2M - \lambda_{22}) \sin \varphi + \alpha \lambda_{33} \cos \varphi] \Omega R_e d\varphi / dt &= \lambda_{33} \Omega \cos \varphi dh / dt \\ (M + \lambda_{22}) R_e d^2 \varphi / dt^2 + (2M + \lambda_{11}) \Omega R_e \sin \varphi \cos \varphi d\lambda / dt &= -[M - m(h)] g \alpha \\ \lambda_{33} [d^2 h / dt^2 + R_e \alpha d^2 \varphi / dt^2] + \lambda_{11} \Omega R_e \cos^2 \varphi d\lambda / dt &= [M - m(h)] g \end{aligned} \tag{2.8}$$

where φ and λ are the latitude and longitude of the centre of masses of the disc, h is the depth of immersion of the centre of mass, α is the angle of inclination of the disc to the water level, $m(h)$ is the mass of the forced out liquid and R_e is the radius of the Earth.

In the approximation of the precession theory of gyroscopes,^{5,15} the first two equations of motion of the centre of mass of the disc, obtained from the initial equations by truncation inertia terms and valid for long periods of time, take the form

$$\begin{aligned} -[(2M - \lambda_{22}) \sin \varphi + \alpha \lambda_{33} \cos \varphi] \Omega R_e d\varphi / dt &= \lambda_{33} \Omega \cos \varphi dh / dt \\ (2M + \lambda_{11}) \Omega R_e \sin \varphi \cos \varphi d\lambda / dt &= -[M - m(h)] g \alpha \end{aligned} \tag{2.9}$$

For real parameters of rings $|\alpha \lambda_{33} \cos \varphi| \gg (2M - \lambda_{22}) \sin \varphi$, and therefore from the first equation we have the equation

$$R_e (d\varphi / dt) = -\alpha^{-1} (dh / dt) \tag{2.10}$$

which gives the relation between the vertical displacement of the ring and its horizontal motion along the meridian.

The seawater density ρ depends on the temperature T , salinity s and pressure p .¹ The dependence on pressure is weak, and can be ignored. Then

$$\frac{d\rho}{dt} = -\frac{\partial \rho}{\partial T} \frac{dT}{dt} + \frac{\partial \rho}{\partial s} \frac{ds}{dt}$$

Thus, the sign of the quantity $d\rho/dt$ depends on the ratio of the rates of double diffusion: heat and salt (thermohaline processes).

The direction and velocity of motion depend on whether the equilibrium position is higher or lower than the centre of mass of the ring, and on the influence on its density of the diffusion of heat and salt. Curves of the dependence of the density of Gulf Stream waters and neighbouring waters on the depth practically repeat the curves of the dependence of the temperature on depth (for example, <http://kingfish.coastal.edu/marine/gulfstream/p2a.htm>), which gives grounds for assuming that the density of the neighbouring waters depends chiefly on the temperature and depends very little on the salinity.

The diffusion of heat seems to affect the density of Gulf Stream rings more strongly than does the diffusion of salt, i.e., the following inequality is satisfied

$$\left| \frac{\partial \rho_r dT}{\partial T dt} \right| > \left| \frac{\partial \rho_r ds}{\partial s dt} \right|$$

A cyclonic ring is cold, and, as a result of heating by radiative heat flow through the free surface and by surrounding warm waters, its density seems to change more with the temperature than with the salinity, i.e.,

$$\left| \frac{\partial \rho_r dT^0}{\partial T^0 dt} \right| > \left| \frac{\partial \rho_r ds^0}{\partial s^0 dt} \right| \tag{2.11}$$

where T^0 is the average temperature, and s^0 is the average salinity of the ring. Its density is reduced, and it surfaces, more correctly, the position of its centre of mass rises ($dh/dt < 0$). If this occurs non-quasi-statically ($M \approx m(h)$) but with delay in relation to the reduction in its density ($M < m(h)$), the centre of mass of the ring lies below the equilibrium position. A disc modelling a cyclonic ring is inclined to the north ($\alpha < 0$) in the meridian plane. On account of the large associated masses of the disc in a direction perpendicular to its plane, its velocity of motion in the meridian plane is directed along the line of intersection of the meridian plane and the plane of the disc. The projection of the velocity in the meridian plane, because of surfacing of the ring, has a component directed to the south and equal to the vertical projection of the velocity of the disc divided by the tangent of the small angle of inclination of its plane to the water level (2.10).

For motion along the parallel, geostrophic balance occurs: the projection onto the tangential plane of the active force opposite to the rolling force is directed towards the south and is balanced by the Coriolis force directed towards the north. Here, the ring has a velocity component directed to the west, determined from the second equation of system (2.9).

Taking relation (1.14) into account, we have ($\Omega \neq 0$)

$$R_e \cos \varphi \frac{d\lambda}{dt} = \frac{M - m(h)}{(2M + \lambda_{11})N^2} g \omega \text{ctg} \varphi \quad (2.12)$$

With the values of the parameters

$$\rho_f \sim 27 \text{ y.e.}, \quad d\rho_f/dz \sim -1.3 \cdot 10^{-3} \text{ y.e./M}, \quad N^2 \sim 1.3 \cdot 10^{-5} \text{ c}^{-1},$$

$$\omega \sim 2 \cdot 10^{-5} \text{ c}^{-1}, \quad \phi \approx 30^\circ, \quad \lambda_{11} \approx M, \quad \frac{M - m(h)}{M} \approx -10^{-3}$$

we obtain $R_e \cos \varphi (d\lambda/dt) \approx -1.8 \text{ cm/s}$. This is similar to the value of the western component of the velocity of motion of the cyclonic ring ($\approx -2 \text{ cm/s}$) according to observational data.^{1,7}

Thus, the resultant velocity vector of the motion of cyclonic rings of the Gulf Stream is directed to the south-west.

An anticyclonic ring of the Gulf Stream is warm, is cooled by the surrounding cold waters and, as a result of this cooling, under condition (2.11), sinks ($dh/dt > 0$). The centre of mass of this ring lies above the equilibrium position. A disc modelling an anticyclonic ring in the meridian plane is inclined to the south ($\alpha > 0$).⁵ Like to the case of a cyclonic ring, the projection of the velocity of motion of the disc in the meridian plane is directed along the line of intersection of the meridian plane and the plane of the disc. Because of the submersion of the ring, this projection has a component directed to the south, which is equal to the vertical projection of the velocity of the ring divided by the tangent of the small angle of inclination of the disc to the south (2.10). For projection of the velocity of motion of the ring along the parallel, geostrophic balance occurs: the projection onto the tangential plane of the rolling force is directed to the south and is balanced by the Coriolis force directed to the north. Here, the ring has a velocity component directed to the west. Thus, the resultant velocity vector of horizontal motion of the anticyclonic ring of the Gulf Stream as a whole is directed to the south-west.

The anticyclonic eddies of Oyasio (the Japanese name for the Kuril current, which is a continuation of the Kamchatka current) are formed at the point where Kuroshio and Oyasio merge to the east of Japan. Having formed, these eddies can move against the Oyasio current to the north-east, carrying the warm and saline intermediate Kuroshio water right up to Bussol' Bay and beyond.⁹ Its velocity of motion is relatively low ($\sim 1 \text{ cm/s}$). The Oyasio current is formed by the cold and softer waters of the eastern coast of the Kamchatka Peninsula (in particular, by abundant precipitations, waters of thawing glaciers and fresh waters of the Kamchatka River) and by Pacific Ocean waters. Waters of the Kamchatka River and Kamchatka current and of the Oyasio current itself are supplemented with warm fresh waters of the numerous hot springs of the peninsula and the Kuril islands. This lowers the temperature contrast of the Oyasio anticyclonic eddies and the waters of the current itself that surround them, and consequently reduces the diffusion of heat from the eddies to the surrounding water. The diffusion of salt from the ring ($ds^0/dt < 0$) seems to affect the average density of the ring more strongly than does the diffusion of heat ($dT^0/dt < 0$), i.e. the inverse inequality to (2.11) is satisfied.

From the equation of state it follows that $d\rho_r/dt < 0$, and, from a certain instant, the inequality $(M - m(h)) < 0$ is satisfied, i.e., the Oyasio anticyclonic eddy, unlike the sinking ring of the Gulf Stream, surfaces ($dh/dt < 0$). For an anticyclonic ring $\alpha > 0$, and it follows from Eq. (2.10) that $d\varphi/dt > 0$, i.e., the meridional projection of the velocity of motion of the eddy as a whole is directed to the north, while it follows from Eq. (2.12) that $d\lambda/dt > 0$, i.e., the component of its motion velocity along the parallel is directed to the east. Thus, in accordance with the proposed model, the Oyasio anticyclonic eddy moves to the north-east, which agrees with data of observations.⁹

Thus, the original cause of the motion of rings of oceanic currents is not the β -effect but the difference in the mean density of the ring from the mean density of the surrounding water displaced by it, i.e., the difference in the gravitational force from the Archimedes force, while the nature of motion, in particular its direction, is determined by the angle of inclination of the dynamic equilibrium shape and by the change in the average density of the ring due to thermohaline processes.

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